# Deligne-Lusztig and Schubert Varieties over Classical Groups

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# 1 Introduction and Definitions

Deligne-Lusztig theory was first introduced in 1976 by Pierre Deligne and George Lusztig in their paper *Representations of reductive groups over finite fields.* Their goal was to associate algebraic varieties to any finite group of Lie type and using them to construct the representations of such a finite group. This was achieved by Lusztig in 1985 who found all representations of all finite simple groups of Lie type.

Their work is a generalization of Vladimir Drinfeld's work on the discrete series representations of $SL_2(\mathbb{F}_q)$. They took the affine curve $X$ and generalized it to a $T^F$ bundle over a Deligne-Lusztig Variety, where $T \subset G$ is a maximal torus.

In this paper we will introduce both Deligne-Lusztig varieties and Schubert varieties over some classical groups. We will also discuss a proof of affiness for Deligne-Lusztig varieties.

## 1.1 Schubert Varieties

Let $T \subset G$ be the maximal torus. Then, $Z(T)^o$ is called the **Cartan subgroup** of $G$. Some facts about this group is that $Z(T) = Z(T)^o$, all Cartan subgroups are conjugate to each other, and that $Z(T)$ is a nilpotent group. Let $B$ be a Borel subgroup such that $T \subset Z(T) \subset B$. Then, we have the following theorem.

**Theorem 1.** *The canonical map*

$$Z[T]\backslash N[T]/Z[T] = W \to B\backslash G/B$$

*is a bijection.*

**Corollary 1.** *This gives us that*

$$\begin{aligned} G/B &= \coprod_{w \in W} X_w \qquad B\text{-orbits on } G/B \\ G/B \times G/B &= \coprod_{w \in W} Y_w \qquad G\text{-orbits on } G/B \times G/B \text{ via diagonal action} \end{aligned}$$

**Definition 1.** *The* ***Bruhat order*** *on $W$ is given by $w' \leq w$ if $\overline{X}_w \supset X_w'$. The closure*

$$\overline{X}_w = \bigsqcup_{w' \leq w} X_w'$$

*is called the* ***Schubert variety****.*

In more detail, the construction is given as follows. Let $w \in W$, where $W$ is the Weyl group, then there exists a coset $wQ$ in $G/Q$, where $Q$ is the parabolic subgroup. This coset will be denoted by $e_{w,Q}$. The set of $T-$ fixed points in $G/Q$ given by the action of left multiplication will be

$$\{e_{w,Q} : w \in W_Q^{\min}\}.$$

Where $W_Q$ is the Weyl group of $Q$ and $W_Q \cong N_Q(T)/T$. $W_Q^{\min}$ is then the set of minimal representatives of $W/W_Q$. Take $w \in W_Q^{\min}$ and let $X_Q(w)$ be the Zariski closure of $Be_{w,Q}$ in $G/Q$. Then, $X_Q(w)$ with the canonical reduced scheme structure is the **Schubert variety** in $G/Q$ that is associated with $wWQ$. Then, $X(w)$ is the Schubert variety in $G/Q$ that is indexed by $w \in W$ when $Q = B$.

## 1.2 Deligne-Lusztig Varieties

Let $G$ be a connected reductive group over $\mathbb{F}_q$ with a Frobenius map $F : G \to G$. Then, we consider $T \subset B$ where $T \subset G$ is the maximal torus over $\mathbb{F}_q$ and $B = TU$ is a Borel subgroup contining $T$ with unipotent radical. Consider the variety

$$\{g \in G : g^{-1}F(g) \in F(U)\} = \mathcal{L}^{-1}(F(U)) \subset G$$

where $g^{-1}F(g) = Y_{T\subset B}$ and we have the following maps

$$\begin{tikzcd}
\mathcal{L}^{-1}(F(U)) \arrow[d] & \subset & G \arrow[loop above, "G^F - \text{torsor}"] \arrow[d, "\mathcal{L}"] & g \arrow[d] \\
F(U) & \subset & G & g^{-1}F(g)
\end{tikzcd}$$

**Lemma 1.** *There exists a $G^F \times T^F$ action on $Y_{T\subset b}$ given by $(ht)y = hyt^{-1}$.*

*Proof.* We first have that

$$(hg)^{-1}F(hg) = g^{-1}h^{-1}F(h)F(g) = g^{-1}F(g) \in F(U).$$

We also have that

$$\begin{aligned}(gt^{-1})^{-1}F(gt^{-1}) &= tgF(g)F(t)^{-1} = F(t)gF(g)F(t)^{-1} \\ &= F(t)gF(g)F(t^{-1} \in F(tUt^{-1})) = F(U).\end{aligned}$$

□

**Lemma 2.** *There exists a right $U \cap F(U)$ action on $Y_{T\subset B}$ given by $g \cdot u = gu$.*

*Proof.* $g(u)^{-1}F(gu) = u^{-1}g^{-1}F(g)F(u) \in F(u)$ since $U^{-1}F(u) \in F(U)$. □

**Definition 2.** *The quotient varieties given by*

$$\begin{aligned}\tilde{X}_{T\subset B} &= \mathcal{L}^{-1}(F(U))/U \cap F(U) \\ X_{T\subset B} &= Y_{T\subset B}/T^F = \mathcal{L}^{-1}(F(U))/T^F(U \cap F(U))\end{aligned}$$

*are called the **Deligne-Lusztig varieties**.*

# 2 Example of Deligne-Lusztig Varieties for Classical Groups

Take $V$ to be an $n-$dimensional vector space over $k$ and let $G = GL(V)$. Let $b = (b_1, ..., b_n)$ be a basis of $V$ and let $T^*$ be the group of diagonal matrices and $B^*$ be the group of upper triangular matrices. Then, the Weyl group will lift into a subgroup of $N(T^*)$ that consists of the $w's$ inducing a permutation of basis vectors. Then, we have the following descriptions

1. The maximal torus $T$ will be $T = G_m^n$ and $W = \mathcal{G}_n$. The action of $W$ on $T$ is given by permutation.

2. A **flag** in $V$ of length $s$ is a sequence of distinct subspaces where $V_0 = \{0\}$ and $V_0 \neq V_1 \subset V_2 \subset \cdots \subset V_s$. A flag is called a **complete flag** if $s = n$. The set of all Borel subgroups, $X$, is then the space of complete flags $V_1 \subset \cdots \subset V_{n-1}$.

3. We have that the quotient $E = G/U^*$ is then the space of complete flags which are marked by non-zero vectors $e_i \in V_i/V_{i-1}$. $T$ will then act on $E$ by $(V', (e_i))(\lambda_i) = (V', (\lambda_i e_i))$. This gives us a $G$-equivariant $T-$torsor over $X$.

4. If $V'$ and $V''$ are two flags, then their relative position is labelled by the permutation $w$ such that $Gr^{V'}_{w(i)} Gr^{V''}_i(V) \neq 0$. Additionally, the following isomorphisms

   $$Gr^{V'}_{w(i)} \cong Gr^{V''}_i Gr^{V'}_{w(i)}(V) \cong Gr^{V'}_{w(i)} Gr^{V''}_i(V) \cong Gr^{V''}_i(V)$$

   will induce a $w-$isomorphism between the $T-$torsor $E(V')$ of markings of $V'$ and $E(V") : e \mapsto e \cdot w$.

Now, take $k$ to be the algebraic closure of the prime field $\mathbb{F}_q$ and suppose that $V$ has an $\mathbb{F}_q$ structure. For $w = (1, \ldots, n)$, for a falg $V'$ to be in relative position $w$ with its image under the Frobenius is that $V'$ is the flag

$$V_1 \subset V_1 + FV_1 \subset V_1 + FV_1 + F^2 V_1 \subset ...$$

and $V = \oplus_0^{n-1} F^i V_1$. Let $P(V)$ be the set of homogeneous lines in $V$. Then the map $V' \mapsto V_1$ is an isomorphism from $X(w)$ to the set of all $x \in P(V)$ that do not lie on an $\mathbb{F}_q$-rational hyperplane. A marking $e$ of $F$ is such that $F(e) = e \cdot w$ if and only if

$$\begin{aligned}
&e_2 \equiv F(e_1)(\mod e_1)\\
&e_3 \equiv F^2(e_1)(\mod e_1, F(e_1)), \ldots\\
&e_n \equiv F^{n-1}(\mod e_1, F(e_1), \ldots, F^{n-1}(e_1 1))\\
&e_1 \equiv F^n(e_1)(\mod F(e_1), \ldots, F^{n-1}(e_1)).
\end{aligned}$$

We have that $e$ is defined by $e_1 \in V_1$ and has the following condition

$$e_1 \wedge F(e_1) \wedge \cdots \wedge F^{n-1}(e_1) = F^n(e_1) \wedge F(e_1) \cdots \wedge F^{n-1}(e_1).$$

If we are given that $(x_i)$ are the coordinates of $e_1$ with respect to some rational basis this can be rewritten as

$$(-1)^{n-1}(\det(x_i^{q^{j-1}}))^{q-1} = 1$$

for $1 \leq i, j \leq n$. This form is invariant under $GL(n, \mathbb{F}_q)$. Up to scalars, this is the product of all non-zero $\mathbb{F}_q-$rational linear forms. The map $(V', e) \mapsto e_1$ induces an isomorphism of $\tilde{X}(w)$ with the affine hypersurface. This hypersurface is stable under the action of $T(w)^F$, which is given by $x \mapsto \lambda x$ where $\lambda \in \mathbb{F}_{q^n}^*$.2

# 3 Example of Schubert Varieties for Classical Groups

## 3.1 The Grassmanian

Let $V = k^n$. **The Grassmanian** $G_{d,n}$ is the set of all $d-$dimensional subspaces $U \subset V$. Let $U$ be an element of $G_{d,n}$ and $b_1, ..., b_d$ a basis of $U$. Then, this basis gives us an $n \times d$ matrix. We can then identify an $n \times d$ matrix with a point in the affine space $\mathbb{A}^{nd}$ and see that $G_{d,n}$ can be viewed as $(\mathbb{A}^{nd} \backslash Z)/\sim$ where $Z$ is the set of $n \times d$ matrices of rank less than $d$ and $\sim$ is defined by $A \sim A'$ if there exists $C \in GL_d(K)$ such that $A' = AC$.

We define the following set

$$I_{d,n} = \{i = (i_1, ..., i_d) \in \mathbb{Z} : 1 \leq i_1 < ... < i_d \leq n\}.$$

$I_{d,n}$ has $N = \binom{n}{d}$ elements and the coordinates of the affine space $\wedge^d V = k^N$ is indexed by $I_{d,n}$. Suppose that $X = V \oplus \cdots \oplus V = k^{nd}$. The **Pücker map** is induced by the exterior product map and is given by

$$p : G_{d,n} \to \mathbb{P}(\wedge^d V) = \mathbb{P}^{N-1}.$$

This map is injective and defines a projective variety structure on $G_{d,n}$.

## 3.2 Schubert varieties of $G_{d,n}$

Let $e_1, ..., e_n$ be the standard basis of $V$. Let $V_i$ be the subspace spanned by $\{e_1, ..., e_i\}$ for $1 \leq i \leq n$. For each $i \in I_{d,n}$ the Schubert variety in $G_{d,n}$ associated to $i$ is defined to be

$$X_i = \{U \in G_{d,n} : \dim(U \cap V_{i_t}) \geq t, 1 \leq t \leq d\}.$$

Define a partial order on $I_{d,n}$ by $i \geq j$ if and only if $i_t \geq j_t$ for all $1 \leq t \leq d$. Suppose that $X_i, X_j$ are associated Schubert varieties in $G_{d,n}$; we have that

$i \geq j$ if and only if $X_i \supseteq X_j$. This means that the partial order $\geq$ on $I_{d,n}$ is induced by the Bruhat-Chevalley order on the set of Schubert varieties.

Let $G = SL(n)$ and let $T$ be the maximal torus in $G$ that consists of diagonal matrices. Then, $W$ can be identified with the symmetric group $S_n$. Let $B$ denote the Borel subgroup of $G$ that consists of upper triangular matrices. For the action of $G$ on $\mathbb{P}(\wedge^d V)$ the $T-$fixed points are precisely the $T-$eigenvectors in $\wedge^d V$. We have the following decomposition as $T-$modules

$$\wedge^d V = \bigoplus_{i \in I_{d,n}} Ke_i.$$

This means that the $T-$fixed points in $\mathbb{P}(\wedge^d V)$ are exactly $[e_i]$, $i \in I_{d,n}$ and these points belong to $G_{d,n}$. Then, the Schubert variety $X_i$ associated to $i$ can also be viewed as the Zariski closure of the $B-$orbit $B[e_i]$ through the $T-$fixed point $[e_i]$ with the canonical reduced scheme structure.

# 4 Proof of Affiness for Deligne-Lusztig Varieties

The theorem that we want to prove is given by

**Theorem 2.** *If there exists* $\mu \in X(T^*) \otimes \mathbb{R}$ *that satisfies* $\mu \in D^0(C, -w^{-1}C)$ *and* $F^*\mu - w\mu \in C^0$ *then* $X(w)$ *is affine. As a consequence,* $X(w)$ *is affine as soon as* $q$ *is larger then the Coxeter number* $h$ *of* $G$.

Let $X(T^*)$ be the character group of $T^*$ and let $C \subset X(T^*) \otimes \mathbb{R}$ be the fundamental chamber. $C_1$ and $C_2$ are two chambers and we denote $D(C_1, C_2)$ to be the intersection of the closed radicial half spaces containing $C_1$ and $C_2$ and denote $D^0(C_1, C_2)$ to be its interior.

**Proposition 1.** *Let* $\overline{O(w)}'$ *be the noramlization of the closure* $\overline{O(w)}$ *of* $O(w)$ *in* $X \times X$. *The map* $\Psi : (w) : pr_1^* E_\lambda \to pr_2^* E_{w^{-1}(\lambda)}$ *extends over* $\overline{O(w)}'$ *if and only if* $\lambda \in D(C, -wC)$. *It vanishes outside of* $O(w)$ *if and only if* $\lambda \in D^0(C, -wC)$.

Now, we assume that $T^*$ and $B^*$ are $F-$stable. This means that the identification of $\sigma(T^*, B^*)$ of $T^*$ and $N(T^*)/T^*$ wth the torus $T$ and the Weyl group $W$ is then compatible with $F$. Then, with these assumptions this means that we have

$$F^* E_\lambda = E_{F*\lambda}$$

where $F^\star\lambda = \lambda \circ F$. The line bundle $pr_1^* E_\lambda^{-1} \otimes pr_2^* E_{w^{-1}(\lambda)}$ is isomorphic to $pr_1^*(E_{F^* w^{-1}\lambda - \lambda})$.If $\lambda \in D^0(C, -wC)$ the section $\Psi(w)$ of it has a zero set that is the complement of $X(w)$ in the projective variety $\overline{X(s)}$. If both of these conditions are fulfilled then $X(w)$ is affine. If we set $\mu = -w^{-1}(\lambda)$ these are exactly the conditions that appear in Theorem 2. Now it remains to be shown that if $q \geq h$ then some $\mu$ will satisfy these two conditions. The first condition, $\mu \in D(C, -w^{-1}C)$ is fulfilled if $\mu \in C^0$. This means that $\langle \mu, H_a \rangle > 0$ for each simple root $a$. We take $\mu$ so that $\langle \mu, H_a \rangle = 1$ for each simple root $a$. Then, we have that $\langle F\mu, H_a \rangle = q$ and $\langle w\mu, H_a \rangle = \langle \mu, wY^{-1} H_a \rangle$. If $H = \Sigma n_a H_a$ is the highest coroot, then $\Sigma n_a = h - 1$ and

$$\langle F\mu - w\mu, H_a \rangle \geq q - \langle \mu, H \rangle = q - h + 1 > 0$$

which gives us the second condition, $F^* \mu - w\mu \in C^0$. Thus, this implies that $X(w)$ is affine.

## 5 References

Most of Section 1 is taken from Prof. Chen's notes on DL theory. Some content from the introdution to Schubert varieties is taken from chapter 2 of Billey and Lakshmibai's *Singular Loci of Schubert Varieties.* Section 2 is mainly from section 2 of Deligne and Lusztig's *Representations of Reductive Groups over Finite Fields.* Section 3 is from chapter 3 of Billey and Lakshmibai's *Singlular Loci of Schubert Varieties.* The last section was taken from section 9 of Deligne and Lusztig's *Representations of Reductive Groups over Finite Fields.*